Classifying Isolated Symmetries

a decades-old problem solved

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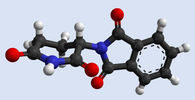
Symmetry Everywhere

from chemistry to cryptography

Symmetry plays a fundamental role across all sciences.

It is symmetry that …



explains the tragic side-effects of the **drug thalidomide** seen in the 1960s.

provides the language for the **Standard Model** in theoretical physics.

is at the heart of novel **cryptography** that is secure in our era of **quantum computers**.

Group Theory

the mathematics of symmetry

Group theory is the area of mathematics dedicated to general properties of symmetry that apply to **all contexts**, including 3-dimensional molecules, 196883-dimensional physics concepts and abstract objects that have no geometric interpretation.

A fruitful way to study an object is to consider the **group** of all its symmetries. This began in the 1800s with Évariste Galois seeing how to solve equations via their symmetries. Today this perspective is **ubiquitous in mathematics and physics**.

We join two symmetries by a yellow line if by carrying out combinations of these two symmetries, one after the next, we can **obtain all the symmetries** of the shape.

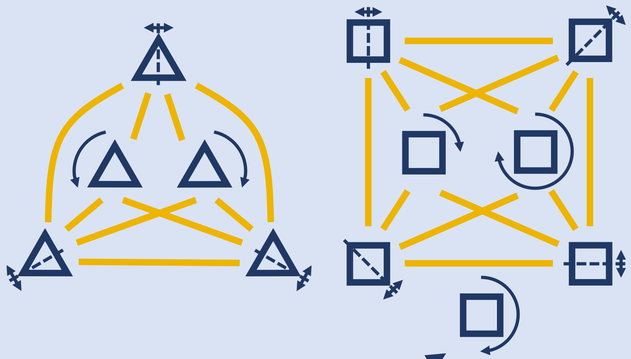
reflection in vertical — rotation by 120°

These are **joined** as each symmetry is some combination of them.

rotation by 90° — rotation by 270°

These are **not joined** as combining rotations can never give a reflection.

symmetries of a triangle **symmetries of a square**



an isolated symmetry
a symmetry not joined to any other symmetry

One of the first things you might notice is that the square has an isolated symmetry but the triangle does not.

In fact, a regular polygon has no isolated symmetries if and only if it has a prime number of corners.

What about isolated symmetries in more general contexts?

Recent Breakthrough

a complete classification

In 1975, Brenner and Wiegold naturally asked:

which symmetry groups have an isolated symmetry?

We answered this **longstanding question**.

In fact, we showed that a symmetry group has an isolated symmetry exactly when it has a noncyclic quotient, and these latter groups are easy to classify.

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Applications

computing random symmetries

Researchers who exploit symmetry use **computer algorithms** to carry out massive calculations involving groups of symmetries.

We can **store symmetries efficiently** on a computer if we know all the symmetries can be obtained by combining just two of them.

Our work sheds light on the mystery of why the **random symmetry generator** (called the product replacement algorithm) works.

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> Random(G)
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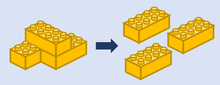
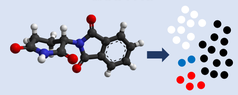
Our Methods

the building blocks of symmetry

Just as …

Lego buildings break into Lego bricks	whole numbers break into prime numbers	molecules break into atoms
	$30 \rightarrow 2 \times 3 \times 5$	

… groups of symmetries break into indivisible **simple groups**.

Classifying all the simple groups, the **atoms of symmetry**, was one of the **greatest mathematical achievements** of the last century.

Lego Principle: If all your Lego bricks are yellow, then anything you build from them will be also yellow.

Similarly, you can deduce facts about a group from facts about the simple groups it is built up from.

Using **numerous branches of mathematics** (such as probability, algebra, geometry, combinatorics), we created new tools for simple groups, which have already been **used by other researchers**.