Eigenvalue Bounds for Perturbed Periodic Dirac Operators

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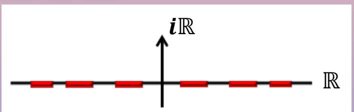
The Background

Consider the Dirac operator describing a relativistic electron in one dimension, $H = H_0 + V$ as an operator in $L^2(\mathbb{R})^2$,

$$H = -i\sigma_2 \frac{d}{dx} + m\sigma_3 + q(x) + V(x)$$

where σ_2, σ_3 are Pauli matrices, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $m>0$ is the particle mass, and the particle is moving in a force field whose potential is composed of two parts $q(x) + V(x)$, where $q(x)$ is real-valued and periodic of period a, and $V(x)$ is a 2 × 2 matrix-valued function with integrable entries in $L^1(\mathbb{R})$.

The periodic Dirac operator H_0 is self-adjoint and its spectrum has a band-gap structure, which means it is purely continuous and consists of a sequence of closed intervals on the real line.



The spectrum $\sigma(H_0)$

H_0 provides a simple model for electron movement in a crystal. Its spectrum is the set of admissible electron energies and is connected to the question whether the crystal is a conductor or insulator.

The Project: Aim and Results

The Dirac operator $H = H_0 + V$ can have additional eigenvalues (bound state energies) in the complex plane.

This project aims to establish a priori regions in the complex plane which contain all eigenvalues of H.

As a first result, we have proved that a complex λ is in the exclusion zone (i.e. cannot be a bound state energy) if

$$\|V\|_1 < \Gamma\big(M(\lambda)\big)\, \gamma_+(\lambda)\, \gamma_-(\lambda),$$

where $M(\lambda)$ is the transfer matrix of the periodic problem (H_0) across one period interval of $q(x)$ and Γ is a matrix function related to the angle between the eigenvectors of the matrix; $\gamma_+(\lambda)$, $\gamma_-(\lambda)$ relate to the size of solution of the periodic problem (H_0).

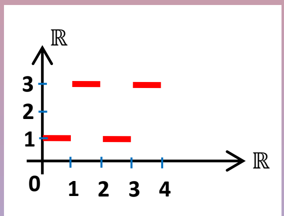
The aim of identifying the bounding regions thus leads to two further mathematical questions,

(1) Studying the properties of $\Gamma\big(M(\lambda)\big)$, in particular asymptotics at critical end points of spectral bands;

(2) Estimating $\gamma_+(\lambda)$, $\gamma_-(\lambda)$.

Numerical Example

$\Gamma\big(M(\lambda)\big)$ in the complex plane



The potential q with period a=2.

One can see that for small perturbation V the eigenvalues are confined to a small region around the end-points of spectral bands.